On the expected penalty functions in a discrete semi-Markov risk model with randomized dividends

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Abstract: This paper considers the expected penalty functions for a discrete semi-Markov risk model with randomized dividends. Under the model, individual claims are governed by a Markov chain with finite state space, and the insurer pays a dividend of 1 with a probability at the end of each period if the present surplus is greater than or equal to a threshold value. Recursive formulae and the initial values for the discounted free penalty functions are derived in the two-state model. A numerical example is provided to illustrate the impact of dividend payments on ruin probabilities.

Keywords: expected penalty function; dividend; generating function; recursive formula; semi-Markov risk model

1. Introduction

Survival probability in a semi-Markov risk model was first investigated by Janssen and Reinhard [1], in which the surplus process not only depends on the current state but also on the next state of an environmental Markov chain. Recently, Albrecher and Boxma [2] generalized the approach of Janssen and Reinhard [1] and studied the corresponding discounted penalty function by means of Laplace-Stieltjes transforms. Cheung and Landriault [3] further investigated the problem of Albrecher and Boxma [2] by relaxing some assumptions pertaining to the interclaim time distribution.

For the discrete-time semi-Markov risk model with a restriction imposed on the total claim size, Reinhard and Snoussi [4, 5] derived recursive formulae for the distribution of the surplus just prior to ruin and that of the deficit at ruin in a special case. Chen et al. [6, 7] relaxed the restriction of Reinhard and Snoussi [4, 5] and derived recursive formulae for computing the expected discounted dividends and survival probabilities for the model. As was mentioned in Chen et al. [7], the discrete-time semi-Markov risk model without

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restriction embraces some existing discrete-time risk models including the compound binomial model (with time-correlated claims) and the compound Markov binomial model (with time-correlated claims) which have been extensively studied by various authors; see, for example, Cossette et al. [8, 9], Yuen and Guo [10], Xiao and Guo [11] and references therein. This motivates us to carry out further ruin analysis for the discrete-time semi-Markov risk model.

The randomized dividend strategy was studied by Tan and Yang [12], Bao [13], Landriault [14], He and Yang [15], and Yuen et al. [16], for the compound binomial model. Under this dividend payment strategy, the insurer pays a dividend of 1 with probability $1-\alpha$ when the surplus is greater than or equal to an arbitrary given non-negative integer x. In this paper, we incorporate randomized dividends into the discrete-time semi-Markov risk model of Chen et al. [6, 7], and examine the corresponding discounted free Gerber-Shiu penalty function.

The rest of the paper is organized as follows. In Section 2, we present the mathematical formulation of the discrete semi-Markov model with randomized dividends. In Section 3, we derive recursive formulae for computing the discounted free Gerber-Shiu penalty function for the model. In Section 4, we obtain two important equations for determining the required initial values when applying the recursive formulae. Sections 5 and 6 are devoted to finding the initial values for the case with $x = 0$. Finally, a numerical example is given in Section 7.

2. The risk model

Let $(J_n, n \in \mathbb{N})$ be a homogeneous, irreducible and aperiodic Markov chain with finite state space $M = \{1, \ldots, m\}$ $(1 \leq m < \infty)$. Its one-step transition probability matrix is given by

$$\mathbf{P} = (p_{ij})_{i,j \in M}, \qquad p_{ij} = \mathbb{P}(J_n = j | J_{n-1} = i, J_k, k \leq n-1),$$

with a unique stationary distribution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m)$. The insurer's surplus (without paying dividends) at the end of the t-th period, X_t, has the form

$$X_t = u + t - \sum_{i=1}^{t} Y_i, \quad t \in \mathbb{N}_+, \tag{1}$$

where $u \in \mathbb{N}$ is the initial surplus and Y_i denotes the total amount of claims in the i-th period. We further assume that a premium of 1 is received at the beginning of each time period, and that Y_t's are nonnegative integer-valued random variables. The distribution of Y_t's is governed by the environmental Markov chain $(J_n, n \in \mathbb{N})$ in the way that (J_t, Y_t) depends on $\{J_k, Y_k; k \leq t-1\}$ only through J_{t-1}. Define

$$g_{ij}(l) = \mathbb{P}(Y_t = l, J_t = j | J_{t-1} = i, J_k, Y_k, k \leq t-1), \quad l \in \mathbb{N},$$

which describes the conditional joint distribution of Y_t and J_t given the previous state J_{t-1} and plays a key role in the following derivations. Note that $p_{ij} = \sum_{l=0}^{\infty} g_{ij}(l)$. We refer the readers to Reinhard and Snoussi [4, 5] for more details about the model.

We now modify the surplus process (1) by allowing dividend payments. Specifically, we assume that the insurer will pay a dividend of 1 with probability $1-\alpha$ at the end of each period if the present surplus is greater than or equal to a threshold value $x \in \mathbb{N}$. Then the modified surplus at the end of the t-th period is given by

$$U_t = u + t - \sum_{i=1}^{t} Y_i - \sum_{i=1}^{t} \gamma_i \mathbf{1}_{(U_{i-1} \geq x)}, \qquad t \in \mathbb{N}_+, \tag{2}$$

where $\mathbf{1}_A$ is the indicator function of event A and γ_i is a series of i.i.d. random variables that are independent of Y_i with $\mathbb{P}(\gamma_i = 0) = \alpha > 0$ and $\mathbb{P}(\gamma_i = 1) = 1 - \alpha$.

Let $\tau = \inf\{t \in \mathbb{N}_+ : U_t < 0\}$ be the time of ruin. The Gerber-Shiu expected discounted penalty function given the initial surplus u and the initial environment state i is defined as

$$m_i(u) = E(v^{\tau}\omega(U_{\tau-}, |U_\tau|)\mathbf{1}_{(\tau<\infty)}|U_0 = u, J_0 = i), \quad i \in M,\ u \in \mathbb{N}, \tag{3}$$

where $\omega(x, y)$ is a nonnegative bounded function and $0 < v \leq 1$ is the discounted factor. If $v = 1$ and $\omega(x, y) \equiv 1$, then $m_i(u)$ becomes

$$\psi_i(u) = \mathbb{P}(\tau < \infty | U_0 = u, J_0 = i), \quad i \in M,\ u \in \mathbb{N},$$

which is the ultimate ruin probability given the initial surplus u and the initial environment state i. Let $\phi_i(u) = 1 - \psi_i(u)$ be the corresponding survival probability.

For all i and j, we assume that

$$\mu_{ij} = \sum_{k=0}^{\infty} k g_{ij}(k) < \infty,$$

and define

$$\mu_i = \sum_{j=1}^{m} \mu_{ij}, \quad i \in M.$$

Here we consider the following positive safety loading condition for the model

$$\sum_{i=1}^{m} \pi_i \mu_i < 1 - (1 - \alpha) = \alpha,$$

which ensures that ruin is not certain.

In this paper, we only consider the case with $v = 1$ and $m = 2$. Our aim is to derive a recursive formula for computing $m_i(u)$.

3. Recursive formula for $m_i(u)$

In this section, we derive recursive formulae for $m_i(u)$ for $i = 1, 2$. Let

$$g_i(k) = \sum_{j=1}^{2} g_{ij}(k), \quad \xi_i(u) = \sum_{k=u+2}^{\infty} g_i(k)\omega(u+1, k-u-1),$$
$$\eta_i(u) = \sum_{k=u+1}^{\infty} g_i(k)\omega(u, k-u), \quad i = 1, 2.$$

Note that for $0 \le u < x$ (i.e. $u = 0, 1, \ldots, x-1$), there must be no dividend in the first period; but for $u \ge x$ (i.e. $u = x, x+1, \ldots$), the period may be subject to dividend payment. So we need to distinguish the two cases. For $i = 1, 2$, considering the first time period, we obtain the following equations:

$$m_i(u) = \sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) + \xi_i(u), \quad 0 \le u < x, \tag{4}$$

$$m_i(u) = \alpha\Big[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) + \xi_i(u)\Big] + (1-\alpha)\Big[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_j(u-k) + \eta_i(u)\Big], \quad u \ge x. \tag{5}$$

We now employ the technique of generating functions to derive the recursive formulae for $m_i(u)$. For any function f, its generating function is denoted by $\widetilde{f}$. Then the generating functions of $m_i(u)$ and $g_{ij}(u)$ are given by $\widetilde{m}_i(s) = \sum_{k=0}^{\infty} s^k m_i(k)$ and $\widetilde{g}_{ij}(s) = \sum_{k=0}^{\infty} s^k g_{ij}(k)$, respectively. By multiplying both sides of (4) and (5) by s^{u+1} and summing over u from 0 to ∞, we obtain

$$\begin{aligned}
s\widetilde{m}_i(s) &= \sum_{u=0}^{\infty} m_i(u)s^{u+1} \\
&= \sum_{u=0}^{x-1}\Big[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) + \xi_i(u)\Big]s^{u+1} \\
&\quad + \sum_{u=x}^{\infty}\Big\{\alpha\Big[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) + \xi_i(u)\Big] \\
&\qquad + (1-\alpha)\Big[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_j(u-k) + \eta_i(u)\Big]\Big\}s^{u+1} \\
&= [\alpha + (1-\alpha)s]\sum_{j=1}^{2}\widetilde{g}_{ij}(s)\widetilde{m}_j(s) + \alpha s\widetilde{\xi}_i(s) + (1-\alpha)s\widetilde{\eta}_i(s) \\
&\quad - \alpha\sum_{j=1}^{2} g_{ij}(0)m_j(0) + (1-\alpha)\sum_{u=0}^{x-1} N_i(u)s^{u+1},
\end{aligned}$$

for $i = 1, 2$, where

$$\begin{aligned} N_i(u) &= \sum_{j=1}^{2}\Big[\sum_{k=0}^{u+1} g_{ij}(k)m_j(u+1-k) - \sum_{k=0}^{u} g_{ij}(k)m_j(u-k)\Big] + \xi_i(u) - \eta_i(u) \\ &= \sum_{j=1}^{2}\Big[g_{ij}(0)m_j(u+1) + \sum_{k=0}^{u}[g_{ij}(k+1) - g_{ij}(k)]m_j(u-k)\Big] + \xi_i(u) - \eta_i(u), \end{aligned}$$

for $u = 0, 1, \ldots, x-1$. Let

$$\begin{aligned} &A(\alpha, s) = \alpha + (1-\alpha)s,\ e_i = \sum_{j=1}^{2} g_{ij}(0)m_j(0),\ M_i(s) = \sum_{u=0}^{x-1} N_i(u)s^{u+1}, \\ &H_i(s) = \alpha e_i - \alpha s\widetilde{\xi}_i(s) - (1-\alpha)[s\widetilde{\eta}_i(s) + M_i(s)],\quad i = 1, 2. \end{aligned}$$

Then we have

$$\begin{cases} [A(\alpha, s)\widetilde{g}_{11}(s) - s]\widetilde{m}_1(s) + A(\alpha, s)\widetilde{g}_{12}(s)\widetilde{m}_2(s) = H_1(s), \\ A(\alpha, s)\widetilde{g}_{21}(s)\widetilde{m}_1(s) + [A(\alpha, s)\widetilde{g}_{22}(s) - s]\widetilde{m}_2(s) = H_2(s). \end{cases} \tag{6}$$

It follows from (6) that

$$\begin{aligned} &\left[(A(\alpha, s)\widetilde{g}_{11}(s) - s)\,(A(\alpha, s)\widetilde{g}_{22}(s) - s) - A(\alpha, s)^2\widetilde{g}_{21}(s)\widetilde{g}_{12}(s)\right]\widetilde{m}_1(s) \\ = \ & H_1(s)\,(A(\alpha, s)\widetilde{g}_{22}(s) - s) - H_2(s)A(\alpha, s)\widetilde{g}_{12}(s). \end{aligned} \tag{7}$$

For notational convenience, we define

$$\begin{aligned} &\bar{g}_{ij}(0) = \alpha g_{ij}(0),\ \bar{g}_{ij}(k) = \alpha g_{ij}(k) + (1-\alpha)g_{ij}(k-1),\quad i, j = 1, 2,\quad k \in \mathbb{N}\backslash\{0, 1\}, \\ &\bar{g}_{ii}(1) = \alpha g_{ii}(1) + (1-\alpha)g_{ii}(0) - 1,\quad \bar{g}_{ij}(1) = \alpha g_{ij}(1) + (1-\alpha)g_{ij}(0),\quad i \neq j, \\ &h_i(0) = \alpha e_i,\ h_i(k) = -\alpha\xi_i(k-1) - (1-\alpha)(\eta_i(k-1) + N_i(k-1)),\quad k = 1, 2, \ldots, x, \\ &h_i(k) = -\alpha\xi_i(k-1) - (1-\alpha)\eta_i(k-1),\quad k = x+1, x+2, \ldots,\quad i = 1, 2, \\ &f_k = \sum_{n=0}^{k}\Big[\bar{g}_{11}(n)\bar{g}_{22}(k-n) - \bar{g}_{21}(n)\bar{g}_{12}(k-n)\Big],\quad g_k^{(1)} = \sum_{n=0}^{k} m_1(n)f_{k-n}, \\ &A_k^{(1)} = \sum_{n=0}^{k}\Big[h_1(n)\bar{g}_{22}(k-n) - h_2(n)\bar{g}_{12}(k-n)\Big],\quad k \in \mathbb{N}. \end{aligned}$$

Let $\widetilde{g^{(1)}}(s)$, $\widetilde{f}(s)$ and $\widetilde{A^{(1)}}(s)$ denote the generating functions of $g_k^{(1)}$, f_k and $A_k^{(1)}$ respectively. According to the property of generating function, (7) yields that

$$\widetilde{g^{(1)}}(s) = \widetilde{f}(s)\widetilde{m}_1(s) = \widetilde{A^{(1)}}(s), \tag{8}$$

where $\widetilde{A^{(1)}}(s)$ is the expression on the right hand side of (7). Then comparing the coefficients of s^k on both sides of the above equation gives $g_k^{(1)} = A_k^{(1)}, k \in \mathbb{N}$, that is,

$$\sum_{n=0}^{k} m_1(n)f_{k-n} = A_k^{(1)}, \qquad k \in \mathbb{N}. \tag{9}$$

Similarly one can obtain

$$\sum_{n=0}^{k} m_2(n) f_{k-n} = A_k^{(2)}, \qquad k \in \mathbb{N}, \tag{10}$$

where $A_k^{(2)} = \sum_{n=0}^{k} \Big[- h_1(n)\bar{g}_{21}(k-n) + h_2(n)\bar{g}_{11}(k-n) \Big], \quad k \in \mathbb{N}.$

Similar to Proposition 1 of Chen et al. [7], we have the following result.

Proposition 1. *If both $f_0 = 0$ and $f_1 = 0$, then $\pi_1\mu_1 + \pi_2\mu_2 \geq 1$, which implies that the positive safety loading condition does not hold.*

Proof. Note that

$$\begin{aligned} f_1 = {} & \alpha g_{11}(0)\left[\alpha g_{22}(1) + (1-\alpha) g_{22}(0) - 1\right] + \alpha g_{22}(0)\left[\alpha g_{11}(1) + (1-\alpha) g_{11}(0) - 1\right] \\ & - \alpha g_{21}(0)\left[\alpha g_{12}(1) + (1-\alpha) g_{12}(0)\right] - \alpha g_{12}(0)\left[\alpha g_{21}(1) + (1-\alpha) g_{21}(0)\right] \leq 0. \end{aligned}$$

So if $f_1 = 0$, then

$$\begin{aligned} & g_{11}(0) = 0, \quad g_{22}(0) = 0, \quad g_{21}(0)\left[\alpha g_{12}(1) + (1-\alpha) g_{12}(0)\right] = 0, \\ & g_{12}(0)\left[\alpha g_{21}(1) + (1-\alpha) g_{21}(0)\right] = 0. \end{aligned}$$

In addition, $g_{21}(0)g_{12}(0) = 0$ since $f_0 = \alpha^2\left[g_{11}(0)g_{22}(0) - g_{12}(0)g_{21}(0)\right] = 0$. Hence there are only two possibilities:

(i) $g_{12}(0) = 0$ and $g_{21}(0) \neq 0$;
(ii) $g_{21}(0) = 0$ and $g_{12}(0) \neq 0$.

Since the remaining steps of the proof are the same as those in Chen et al. [7], we omit the details here. □

Finally, from (9), (10) and Proposition 1, we obtain the following recursive formulae

$$m_i(k) = \begin{cases} \frac{1}{f_0}\left[A_k^{(i)} - \sum_{n=0}^{k-1} m_i(n) f_{k-n}\right], & \text{if } \ f_0 \neq 0, \\ \frac{1}{f_1}\left[A_{k+1}^{(i)} - \sum_{n=0}^{k-1} m_i(n) f_{k+1-n}\right], & \text{if } \ f_0 = 0 \ \text{ and } \ f_1 \neq 0, \end{cases} \tag{11}$$

for $i = 1, 2$ and $k \in \mathbb{N}_+$.

Remark 1. For $i = 1, 2$ and any $k \in \mathbb{N}$, $A_k^{(i)}$ in (11) involves $\{m_1(u), m_2(u), u = 0, 1, \ldots, \min(k, x)\}$. So we need the values of $m_i(0), m_i(1), \ldots, m_i(x)$ for $i = 1, 2$ when applying the recursive formulae (11). In order to solve for $m_i(0), m_i(1), \ldots, m_i(x)$ for $i = 1, 2$, we need $2(x+1)$ equations. By taking $u = 0, 1, \ldots, x-1$ in (4), we obtain $2x$ equations. The remaining task is to look for another two equations in relation to the $2(x+1)$ values of $m_i(u)$.

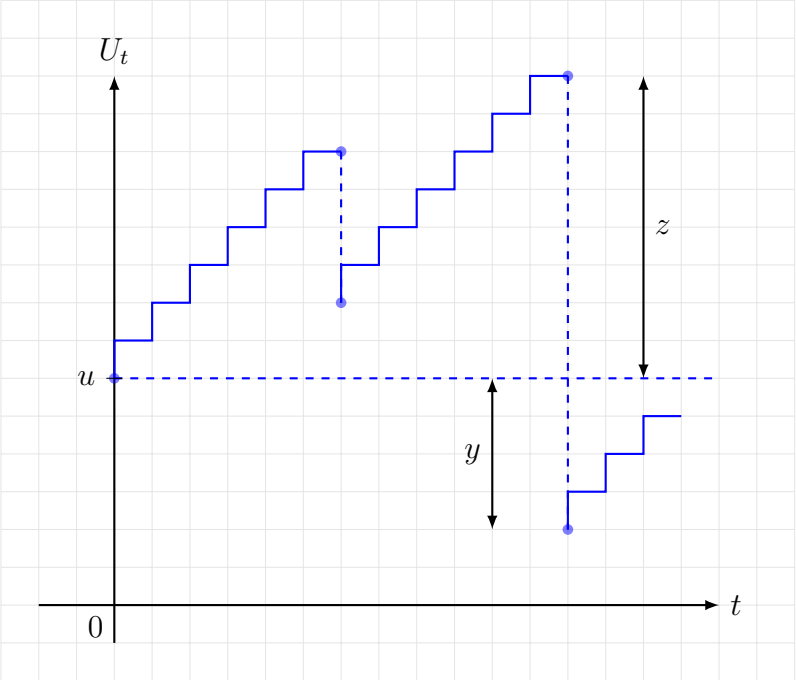

Figure 1: Description of the use of $f_{ij}(0, z, y)$ in the first term of (12)

4. The two equations

Following the work of Ahn and Badescu [17] or Cheung and Landriault [3], one can derive another two equations by conditioning on the first drop below initial surplus. For $u, z = 0, 1, \ldots$ and $y = 1, 2, \ldots$, let

$$f_{ij}(u, z, y) = \mathbb{P}(\tau < \infty, J_\tau = j, U_{\tau-} = z, |U_\tau| = y \mid U_0 = u, J_0 = i), \quad i, j = 1, 2,$$

which is the joint probability function of the surplus immediately before ruin and the deficit at ruin with the initial state being i and the state at ruin being j. As was demonstrated in the aforementioned papers, the joint probability function $f_{ij}(0, z, y)$ plays an important role in deriving another two equations.

Consider a change in the definition of ruin for a moment. If ruin is defined as the first drop below the initial surplus u, then the corresponding joint probability function of the surplus immediately before ruin and the deficit at ruin with the initial state being i and the state at ruin being j is equivalent to $f_{ij}(0, z, y)$ which is defined based on the original definition of ruin. Based on this reasoning, we define

$$m_{ij}(u) = E(\omega(U_{\tau-}, |U_\tau|)\mathbf{1}_{(\tau<\infty, J_\tau=j)}|U_0 = u, J_0 = i), \quad i, j = 1, 2, \ u \in \mathbb{N},$$

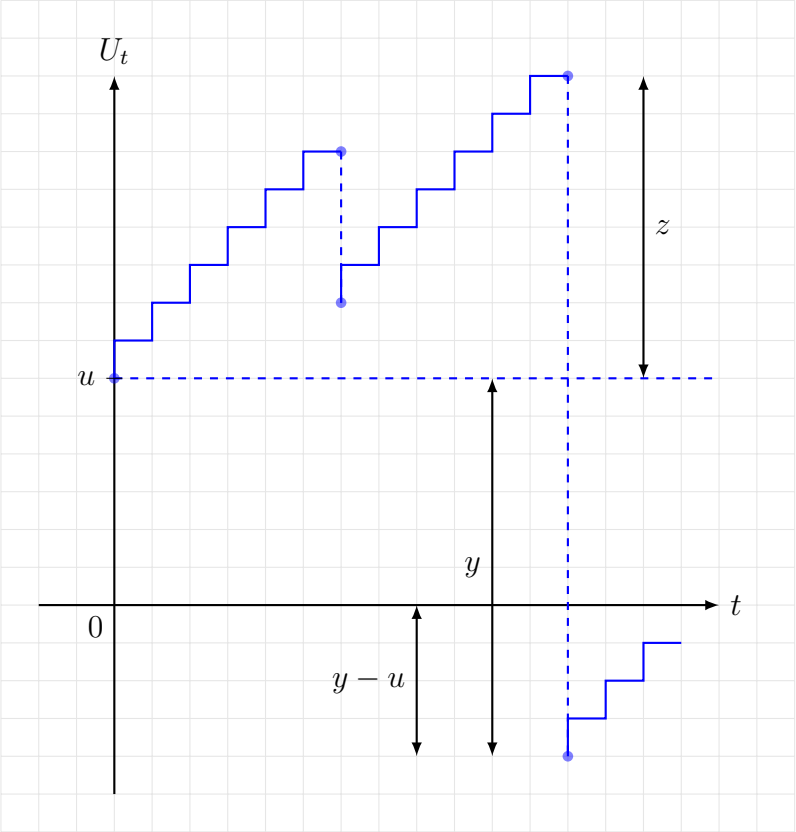

Figure 2: Description of the use of $f_{ij}(0,z,y)$ in the second term of (12)

which can be expressed as

$$\begin{aligned} m_{ij}(u) &= \sum_{k=1}^{2}\sum_{z=0}^{\infty}\sum_{y=1}^{u} f_{ik}(0,z,y)m_{kj}(u-y) \\ &\quad + \sum_{z=0}^{\infty}\sum_{y=u+1}^{\infty} f_{ij}(0,z,y)\omega(u+z,y-u), \quad i,j=1,2, \end{aligned} \tag{12}$$

where the first term on the right side can be explained by Figure 1 and the second term takes care of the situations shown in Figure 2. Specially, taking $u=x$ in (12) yields

$$\begin{aligned} m_{ij}(x) &= \sum_{k=1}^{2}\sum_{z=0}^{\infty}\sum_{y=1}^{x} f_{ik}(0,z,y)m_{kj}(x-y) \\ &\quad + \sum_{z=0}^{\infty}\sum_{y=x+1}^{\infty} f_{ij}(0,z,y)\omega(x+z,y-x), \quad i,j=1,2, \end{aligned}$$

which in turn implies that

$$\begin{aligned} m_{i}(x) &= \sum_{k=1}^{2}\sum_{z=0}^{\infty}\sum_{y=1}^{x} f_{ik}(0,z,y)m_{k}(x-y) \\ &\quad + \sum_{z=0}^{\infty}\sum_{y=x+1}^{\infty} f_{i}(0,z,y)\omega(x+z,y-x), \quad i=1,2, \end{aligned} \tag{13}$$

where $f_i(0,z,y)=\sum_{j=1}^{2} f_{ij}(0,z,y)$. Hence one can use (13) to obtain another two equations which involve $m_i(u)$ for $i=1,2$ and $u=0,1,\ldots,x-1,x$. Then the remaining task is to determine $f_{ij}(0,z,y), i,j=1,2$. On the other hand, note that $m_{ij}(0)=f_{ij}(0,z,y)$ if $\omega(v_1,v_2)=\mathbf{1}_{(v_1=z,\,v_2=y)}$ in (3). So we devote to dealing with $m_{ij}(0)$ for the case with the dividend threshold $x=0$ in the next section.

5. The values of $m_{ij}(0)$ for $x=0$

In this section, we only consider the case with $x=0$. Let

$$\xi_{il}(u)=\sum_{k=u+2}^{\infty} g_{il}(k)\omega(u+1,k-u-1),\quad \eta_{il}(u)=\sum_{k=u+1}^{\infty} g_{il}(k)\omega(u,k-u),\quad i,l=1,2.$$

Corresponding to $e_i, h_i(k), g_k^{(1)}, A_k^{(1)}, A_k^{(2)}$ in Section 3, $e_{il}, h_{il}(k), g_{kl}^{(1)}, A_{kl}^{(1)}, A_{kl}^{(2)}$ can be defined similarly. Then we can find that all of the equations in Section 3 still hold just by making some slight modification. For example, equation (5) can be replaced by

$$\begin{aligned} m_{il}(u)=&\alpha\Big[\sum_{j=1}^{2}\sum_{k=0}^{u+1} g_{ij}(k)m_{jl}(u+1-k)+\xi_{il}(u)\Big]\\ &+(1-\alpha)\Big[\sum_{j=1}^{2}\sum_{k=0}^{u} g_{ij}(k)m_{jl}(u-k)+\eta_{il}(u)\Big],\quad u\in\mathbb{N}. \end{aligned}\tag{14}$$

For notational convenience, we denote $m_{il}(u)$ for a fixed $l(=1,2)$ by $m_i(u)$ with other functions defined similarly in this section. Therefore, in order to calculate $m_1(0)$ and $m_2(0)$, we need to find two equations associated with them.

5.1. The first equation

Note that

$$\lim_{s\to 1}\widetilde{f}(s)=\lim_{s\to 1}\left[\left(A(\alpha,s)\widetilde{g}_{11}(s)-s\right)\left(A(\alpha,s)\widetilde{g}_{22}(s)-s\right)-A(\alpha,s)^2\widetilde{g}_{21}(s)\widetilde{g}_{12}(s)\right]=0.$$

So if $\lim_{s\to 1}\widetilde{m}_1(s)=\sum_{u=0}^{\infty} m_1(u)<\infty$, then it follows from (8) that

$$\begin{aligned} \lim_{s\to 1}\widetilde{A^{(1)}}(s) \;=\;& -p_{21}[\alpha e_1-\alpha\sum_{u=0}^{\infty}\xi_1(u)-(1-\alpha)\sum_{u=0}^{\infty}\eta_1(u)]\\ &-p_{12}[\alpha e_2-\alpha\sum_{u=0}^{\infty}\xi_2(u)-(1-\alpha)\sum_{u=0}^{\infty}\eta_2(u)]=0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &m_1(0)\Big(g_{11}(0)p_{21}+g_{21}(0)p_{12}\Big)+m_2(0)\Big(g_{12}(0)p_{21}+g_{22}(0)p_{12}\Big)\\ =\;& p_{21}\sum_{u=0}^{\infty}\xi_1(u)+p_{12}\sum_{u=0}^{\infty}\xi_2(u)+\frac{1-\alpha}{\alpha}\Big(p_{21}\sum_{u=0}^{\infty}\eta_1(u)+p_{12}\sum_{u=0}^{\infty}\eta_2(u)\Big). \end{aligned}\tag{15}$$

For the case with $\sum_{u=0}^{\infty} m_1(u) = \infty$, we need to check whether (15) still holds? In order to deal with this case, we need the following proposition.

Proposition 2. *For $i = 1, 2$, let $\widetilde{\phi}_i(s)$ denote the generating function of $\phi_i(u)$. Then we have*

$$\lim_{s\to 1} \widetilde{f}(s)\widetilde{\phi}_i(s) = -\widetilde{f}'(1),$$

where $\widetilde{f}'(1) < \infty$ (see Section 5.2 below).

Proof. Using techniques similar to those in Chen et al. [7], one can show that

$$\phi_i(u) = \alpha \sum_{j=1}^{2} \sum_{k=0}^{u+1} g_{ij}(k)\phi_j(u+1-k) + (1-\alpha) \sum_{j=1}^{2} \sum_{k=0}^{u} g_{ij}(k)\phi_j(u-k), \ i = 1, 2, \ u \in \mathbb{N},$$

and

$$s\widetilde{\phi}_i(s) = A(\alpha, s) \sum_{j=1}^{2} \widetilde{g}_{ij}(s)\widetilde{\phi}_j(s) - \alpha \sum_{j=1}^{2} g_{ij}(0)\phi_j(0), \quad i = 1, 2.$$

Let $a_i = \sum_{j=1}^{2} g_{ij}(0)\phi_j(0), \ i = 1, 2$. Then we have

$$\widetilde{f}(s)\widetilde{\phi}_1(s) = \alpha a_1 \left(A(\alpha, s)\widetilde{g}_{22}(s) - s\right) - \alpha a_2 A(\alpha, s)\widetilde{g}_{12}(s). \tag{16}$$

Similar to the derivation of (9), we have

$$\sum_{n=0}^{k} \phi_1(n) f_{k-n} = b_k, \qquad k \in \mathbb{N}, \tag{17}$$

where

$$b_k = \alpha a_1 \bar{g}_{22}(k) - \alpha a_2 \bar{g}_{12}(k) = \alpha(c_k\phi_1(0) + d_k\phi_2(0)),$$

with

$$c_k = g_{11}(0)\bar{g}_{22}(k) - g_{21}(0)\bar{g}_{12}(k), \ d_k = g_{12}(0)\bar{g}_{22}(k) - g_{22}(0)\bar{g}_{12}(k).$$

Let

$$\begin{aligned} &d(0) = \phi_1(0), \qquad d(u) = \phi_1(u) - \phi_1(u-1), \quad u \geq 1, \\ &B(0) = b_0, \qquad B(u) = b_u - b_{u-1}, \quad u \geq 1. \end{aligned}$$

As was proved in Chen et al. [7], we obtain

$$\widetilde{f}(s)\widetilde{d}(s) = \widetilde{B}(s), \quad i = 1, 2, \tag{18}$$

where $\widetilde{d}(s)$ and $\widetilde{B}(s)$ are the generating functions of $d(u)$ and $B(u)$ respectively. Note that

$$\widetilde{d}(1) = \sum_{u=0}^{\infty} d(u) = \lim_{n\to\infty} \phi_1(n) = 1, \quad \widetilde{B}(1) = \lim_{n\to\infty} B(n) = 0, \quad \text{and} \quad \widetilde{f}(1) = 0.$$

As a result, $\widetilde{B}'(1) = \widetilde{f}'(1)$.

On the other hand,

$$\begin{aligned}
\widetilde{B}'(1) &= \sum_{k=1}^{\infty} k(b_k - b_{k-1}) = \sum_{k=1}^{\infty}\sum_{i=1}^{k}(b_k - b_{k-1}) = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty}(b_k - b_{k-1}) \\
&= \sum_{i=1}^{\infty}(-b_{i-1} + \lim_{k\to\infty} b_k) = -\sum_{i=0}^{\infty} b_i = -\alpha\Big(\phi_1(0)\sum_{i=0}^{\infty} c_i + \phi_2(0)\sum_{i=0}^{\infty} d_i\Big) \\
&= \alpha\Big[\phi_1(0)\Big(g_{11}(0)p_{21} + g_{21}(0)p_{12}\Big) + \phi_2(0)\Big(g_{12}(0)p_{21} + g_{22}(0)p_{12}\Big)\Big].
\end{aligned}$$

This together with (16) gives

$$\lim_{s\to 1} \widetilde{f}(s)\widetilde{\phi}_1(s) = -\alpha(a_1p_{21} + a_2p_{12}) = -\widetilde{B}'(1) = -\widetilde{f}'(1).$$

Along the same lines, we have $\lim_{s\to 1} \widetilde{f}(s)\widetilde{\phi}_2(s) = -\widetilde{f}'(1)$. □

Let $\widetilde{\psi}_1(s)$ denote the generating function of $\psi_1(u)$. According to Proposition 2, we have

$$\begin{aligned}
\lim_{s\to 1} \widetilde{f}(s)\widetilde{\psi}_1(s) &= \lim_{s\to 1} \widetilde{f}(s)\Big(\frac{1}{1-s} - \widetilde{\phi}_1(s)\Big) \\
&= \lim_{s\to 1} \frac{\widetilde{f}(s)}{1-s} - \lim_{s\to 1} \widetilde{f}(s)\widetilde{\phi}_1(s) = \widetilde{f}'(1) - \widetilde{f}'(1) = 0.
\end{aligned}$$

Since $\omega(x, y)$ is defined as a non-negative bounded function, we have

$$\lim_{s\to 1} \widetilde{f}(s)\widetilde{m}_1(s) = \lim_{s\to 1} \widetilde{f}(s)\widetilde{\psi}_1(s) = 0,$$

which implies that $\lim_{s\to 1} \widetilde{A^{(1)}}(s) = 0$ by (8). It means that (15) still holds.

Remark 2. Assume that $\omega(x, y) \le K$ for some constant K. Then we see that

$$\sum_{u=0}^{\infty} \xi_i(u) \le K\sum_{u=0}^{\infty}\sum_{k=u+2}^{\infty} g_i(k) = K\sum_{k=2}^{\infty}\sum_{u=0}^{k-2} g_i(k) \le K\mu_i < \infty.$$

Similarly, we have $\sum_{u=0}^{\infty} \eta_i(u) < \infty$.

5.2. The second equation

In this subsection, we use an alternative method to find another relation between $m_1(0)$ and $m_2(0)$. To do it, we consider several cases of $f_0 = \alpha^2[g_{11}(0)g_{22}(0) - g_{12}(0)g_{21}(0)]$.

Case 1. If $f_0 = 0$, it follows from (9) that $f_1 m_1(0) = A_1^{(1)}$, which yields

$$\begin{aligned} & K_1 m_1(0) + K_2 m_2(0) \\ = \ & \alpha^2[g_{12}(0)\xi_2(0) - g_{22}(0)\xi_1(0)] + \alpha(1-\alpha)[g_{12}(0)\eta_2(0) - g_{22}(0)\eta_1(0)], \end{aligned} \tag{19}$$

where

$$\begin{aligned} K_1 &= \alpha^2[g_{22}(0)g_{11}(1) - g_{12}(0)g_{21}(1)] - \alpha g_{22}(0) \le 0, \\ K_2 &= \alpha^2[g_{22}(0)g_{12}(1) - g_{12}(0)g_{22}(1)] + \alpha g_{12}(0) \ge 0. \end{aligned}$$

Furthermore,

$$K_1 = K_2 = 0 \iff g_{12}(0) = g_{22}(0) = 0.$$

In this case, we have $e_1 = g_{11}(0)m_1(0),\ e_2 = g_{21}(0)m_1(0)$, and by (10),

$$\begin{aligned} f_1 m_2(0) = A_1^{(2)} &= \{\alpha^2[g_{21}(0)g_{11}(1) - g_{11}(0)g_{21}(1)] - \alpha g_{21}(0)\}m_1(0) \\ &+ \alpha^2[g_{21}(0)\xi_1(0) - g_{11}(0)\xi_2(0)] + \alpha(1-\alpha)[g_{21}(0)\eta_1(0) - g_{11}(0)\eta_2(0)]. \end{aligned} \tag{20}$$

Case 2. If $f_0 > 0$, then $\widetilde{f}(0) = f_0 > 0$. Note that

$$\begin{aligned} \widetilde{f}'(s) = \ & [(1-\alpha)\widetilde{g}_{11}(s) + A(\alpha,s)\widetilde{g}'_{11}(s) - 1][A(\alpha,s)\widetilde{g}_{22}(s) - s] \\ & + [A(\alpha,s)\widetilde{g}_{11}(s) - s][(1-\alpha)\widetilde{g}_{22}(s) + A(\alpha,s)\widetilde{g}'_{22}(s) - 1] \\ & - 2A(\alpha,s)(1-\alpha)\widetilde{g}_{12}(s)\widetilde{g}_{21}(s) - A(\alpha,s)^2[\widetilde{g}'_{12}(s)\widetilde{g}_{21}(s) + \widetilde{g}_{12}(s)\widetilde{g}'_{21}(s)]. \end{aligned}$$

So we have

$$\begin{aligned} \widetilde{f}'(1) &= -p_{21}[(1-\alpha)p_{11} + \mu_{11} - 1] - p_{12}[(1-\alpha)p_{22} + \mu_{22} - 1] \\ &\quad - 2(1-\alpha)p_{12}p_{21} - p_{21}\mu_{12} - p_{12}\mu_{21} \\ &= -(1-\alpha)(p_{21} + p_{12}) - (p_{21}\mu_1 + p_{12}\mu_2) + p_{21} + p_{12} \\ &= \alpha(p_{21} + p_{12}) - (p_{21}\mu_1 + p_{12}\mu_2) > 0, \end{aligned}$$

where the last inequality follows from the positive safety loading condition. On the other hand, since $\widetilde{f}(1) = 0$, there exists a $\delta > 0$ such that $\widetilde{f}(s) < 0$ for any $s \in (1-\delta, 1)$. As a consequence, there exists a $\rho \in (0,1)$ such that $\widetilde{f}(\rho) = 0$, which in turn implies that $\widetilde{A^{(1)}}(\rho) = 0$, that is,

$$\begin{aligned} & \{[g_{11}(0)\widetilde{g}_{22}(\rho) - g_{21}(0)\widetilde{g}_{12}(\rho)]A(\alpha,\rho) - g_{11}(0)\rho\}m_1(0) \\ & \quad + \{[g_{12}(0)\widetilde{g}_{22}(\rho) - g_{22}(0)\widetilde{g}_{12}(\rho)]A(\alpha,\rho) - g_{12}(0)\rho\}m_2(0) \\ = \ & \rho\Big\{[\widetilde{\xi}_1(\rho) - \frac{1-\alpha}{\alpha}\widetilde{\eta}_1(\rho)][A(\alpha,\rho)\widetilde{g}_{22}(\rho) - \rho] - [\widetilde{\xi}_2(\rho) - \frac{1-\alpha}{\alpha}\widetilde{\eta}_2(\rho)]A(\alpha,\rho)\widetilde{g}_{12}(\rho)\Big\}. \end{aligned} \tag{21}$$

Case 3. If $f_0 < 0$, then $\widetilde{f}(0) = f_0 < 0$. Besides,

$$\begin{aligned}\widetilde{f}(-1) &= [(2\alpha-1)\widetilde{g}_{11}(-1)+1][(2\alpha-1)\widetilde{g}_{22}(-1)+1] - (2\alpha-1)^2\widetilde{g}_{21}(-1)\widetilde{g}_{12}(-1)\\ &> (1-\widetilde{g}_{11}(1))(1-\widetilde{g}_{22}(1)) - \widetilde{g}_{21}(1)\widetilde{g}_{12}(1)\\ &= (1-p_{11})(1-p_{22}) - p_{21}p_{12} = 0, \quad \forall\ \alpha \in (0,1].\end{aligned}$$

So there exists a $\rho \in (-1,0)$ such that $\widetilde{f}(\rho) = 0$, which in turn gives $\widetilde{A^{(1)}}(\rho) = 0$. That is, (21) also holds in this case.

Remark 3. *For Case 2, the method used in Chen et al. [7] is invalid in this paper.*

6. The values of $f_{ij}(0,z,y)$ for $x = 0$

As was shown in Section 4, $f_{ij}(0,z,y)$ with the dividend threshold $x = 0$ plays an important role in the derivation of the equations. So we further consider the joint probability function of the surplus immediately before ruin and the deficit at ruin for the case with $x = 0$ in this section. Since $f_{ij}(0,z,y)$ and $f_i(0,z,y)$ can be derived in a similar way, we just simply demonstrate how $f_i(0,z,y)$ can be obtained for notational convenience.

For $z = 0, 1, \ldots$ and $y = 1, 2, \ldots$, let $\omega(z_1, z_2) = \mathbf{1}_{(z_1=z, z_2=y)}$ in (3). Then

$$m_i(u) = f_i(u,z,y) = \mathbb{P}(\tau < \infty, U_{\tau-} = z, |U_\tau| = y \mid U_0 = u, J_0 = i), \quad i = 1,2,$$

is the joint probability function of the surplus immediately before ruin and the deficit at ruin. For $i = 1, 2$, we have

$$\begin{aligned}\xi_i(u) &= \sum_{k=u+2}^{\infty} g_i(k)\mathbf{1}_{(u+1=z, k-u-1=y)} = g_i(z+y)\mathbf{1}_{(u=z-1)}, \quad u \in \mathbb{N},\\ \eta_i(u) &= \sum_{k=u+1}^{\infty} g_i(k)\mathbf{1}_{(u=z, k-u=y)} = g_i(z+y)\mathbf{1}_{(u=z)}, \quad u \in \mathbb{N},\\ h_i(0) &= \alpha e_i = \alpha\sum_{j=1}^{2} g_{ij}(0)f_j(0,z,y),\\ h_i(u) &= -\alpha\mathbf{1}_{(u=z)}g_i(z+y) - (1-\alpha)\mathbf{1}_{(u=z+1)}g_i(z+y), \quad u \in \mathbb{N}\backslash\{0\}.\end{aligned}$$

Let

$$\theta_k = h_1(0)\bar{g}_{22}(k) - h_2(0)\bar{g}_{12}(k), \quad \beta_k = -h_1(0)\bar{g}_{21}(k) + h_2(0)\bar{g}_{11}(k).$$

Then

$$A_k^{(1)} = \begin{cases} f_0 f_1(0,z,y), & k = 0,\\ \theta_k, & 1 \le k < z,\\ \theta_k - \alpha[g_1(z+y)\bar{g}_{22}(0) - g_2(z+y)\bar{g}_{12}(0)], & k = z,\\ \theta_k - \alpha[g_1(z+y)\bar{g}_{22}(k-z) - g_2(z+y)\bar{g}_{12}(k-z)] & \\ \quad -(1-\alpha)[g_1(z+y)\bar{g}_{22}(k-z-1) - g_2(z+y)\bar{g}_{12}(k-z-1)], & k > z; \end{cases}$$

$$A_k^{(2)} = \begin{cases} f_0 f_2(0,z,y), & k=0, \\ \beta_k, & 1 \le k < z, \\ \beta_k + \alpha[g_1(z+y)\bar{g}_{21}(0) - g_2(z+y)\bar{g}_{11}(0)], & k=z, \\ \beta_k + \alpha[g_1(z+y)\bar{g}_{21}(k-z) - g_2(z+y)\bar{g}_{11}(k-z)] & \\ \quad + (1-\alpha)[g_1(z+y)\bar{g}_{21}(k-z-1) - g_2(z+y)\bar{g}_{11}(k-z-1)], & k>z, \end{cases}$$

and hence, the recursive formula for $f_i(u,z,y)$ follows from (11).

Besides, it is easy to see that

$$\sum_{u=0}^{\infty} \xi_i(u) = g_i(z+y)\mathbf{1}_{(z\ge 1)}, \quad \widetilde{\xi}_i(s) = \sum_{u=0}^{\infty} \xi_i(u)s^u = s^{z-1} g_i(z+y)\mathbf{1}_{(z\ge 1)},$$
$$\sum_{u=0}^{\infty} \eta_i(u) = g_i(z+y), \quad \widetilde{\eta}_i(s) = \sum_{u=0}^{\infty} \eta_i(u)s^u = s^z g_i(z+y), \quad i=1,2.$$

Then the right hand side of (15) becomes

$$[p_{21}g_1(z+y) + p_{12}g_2(z+y)]\Big(\mathbf{1}_{(z\ge 1)} + \frac{1-\alpha}{\alpha}\Big).$$

Finally, the initial values $f_i(0,z,y)$ can be derived from (15) and (19) (or (20) and (21)).

7. A numerical example

In this section, we carry out a numerical example to demonstrate how the proposed method can be applied. In particular, we consider the effect of some parameters on the ruin probability in the example. In order to assess the impact of α and x on the ruin probability, we do the recursive computation for various values of α and x. Specifically, we take $\alpha = 0.8, 0.85, 0.9$ and $x = 0, 1, 2, 3$. Since our analysis does not rely on the form of $g_{ij}(k)$, we arbitrarily choose a simple $g_{ij}(k)$ which coincides with the one used in Reinhard and Snoussi [4, 5]. The distribution of claims $g_{ij}(k)$ is given in Table 1.

Table 1: The distribution of claims.

k	$g_{11}(k)$	$g_{12}(k)$	$g_{21}(k)$	$g_{22}(k)$
0	5/8	0	0	0
1	1/8	1/8	0	1/6
2	1/8	0	1/2	1/6
3	0	0	1/6	0
≥ 4	0	0	0	0

As we have mentioned in Remark 1, in order to apply the recursive formulae (11), we need the values of $\psi_i(0), \psi_i(1), \ldots, \psi_i(x)$ for $i = 1, 2$. Therefore, $2(x+1)$ equations are required for finding the $2(x+1)$ values of ψ's. By taking $u = 0, 1, \ldots, x-1$ in (4),

we obtain $2x$ equations. The other two equations come from (13) where the values of $f_{ij}(0,z,y)$ for $x=0$ play an important role.

We now show how the values of $f_{ij}(0,z,y)$ and $\psi_i(u)$ can be calculated. By direct calculation, we have

$$p_{12}=\frac{1}{8},\quad p_{21}=\frac{2}{3},\quad \mu_1=\frac{1}{2},\quad \mu_2=2.$$

Note that $f_0=0$ and $g_{12}(0)=g_{22}(0)=0$. Then, according to (15) and (20), the values of $\{f_{ij}(0,z,y), i,j=1,2\}$ for $x=0$ can be obtained by the following equations

$$f_{1j}(0,z,y)=\frac{12}{5}\Big[\frac{2}{3}g_{1j}(z+y)+\frac{1}{8}g_{2j}(z+y)\Big]\Big(\mathbf{1}_{(z\geq 1)}+\frac{1-\alpha}{\alpha}\Big),$$
$$f_{2j}(0,z,y)=\frac{6\alpha}{6-\alpha}g_{2j}(1+y)\mathbf{1}_{(z=1)}+\frac{6(1-\alpha)}{6-\alpha}g_{2j}(y)\mathbf{1}_{(z=0)}.$$

With the values of $f_{ij}(0,z,y)$, one can solve the system of $2(x+1)$ equations by using some mathematical software package such as MATLAB to obtain the values of $\psi_i(0),\psi_i(1),\ldots,\psi_i(x)$ for $i=1,2$. Furthermore, the values of $\{N_i(u): i=1,2; u=0,1,\ldots,x-1\}$, $\{A_k^{(i)}: i=1,2; k\in\mathbb{N}\}$ and $\{f_k: k\in\mathbb{N}\}$ can also be obtained. As a result, we can finally use the recursive formulae (11) to calculate all the values of $\psi_i(u)$. In this example, the values of $f_{ij}(0,z,y)$, $i,j=1,2$, are given in Tables 2-5, and some values of the ruin probabilities $\psi_i(u)$ for $x=0,1,2,3$, are listed in Tables 6-9, respectively.

Table 2: The values of $f_{11}(0,z,y)$ for $x=0$.

$f_{11}(0,z,y)$	$y=0$	$y=1$	$y=2$	$y=3$	$y\geq 4$
$z=0$	$\frac{1-\alpha}{\alpha}$	$\frac{1-\alpha}{5\alpha}$	$\frac{7(1-\alpha)}{20\alpha}$	$\frac{1-\alpha}{20\alpha}$	0
$z=1$	$\frac{1}{5\alpha}$	$\frac{7}{20\alpha}$	$\frac{1}{20\alpha}$	0	0
$z=2$	$\frac{7}{20\alpha}$	$\frac{1}{20\alpha}$	0	0	0
$z=3$	$\frac{1}{20\alpha}$	0	0	0	0
$z\geq 4$	0	0	0	0	0

Table 3: The values of $f_{12}(0,z,y)$ for $x=0$.

$f_{12}(0,z,y)$	$y=0$	$y=1$	$y=2$	$y\geq 3$
$z=0$	0	$\frac{1-\alpha}{4\alpha}$	$\frac{1-\alpha}{20\alpha}$	0
$z=1$	$\frac{1}{4\alpha}$	$\frac{1}{20\alpha}$	0	0
$z=2$	$\frac{1}{20\alpha}$	0	0	0
$z\geq 3$	0	0	0	0

Table 4: The values of $f_{21}(0,z,y)$ for $x=0$.

$f_{21}(0,z,y)$	$y=0$	$y=1$	$y=2$	$y=3$	$y\geq 4$
$z=0$	0	0	$\frac{3(1-\alpha)}{6-\alpha}$	$\frac{1-\alpha}{6-\alpha}$	0
$z=1$	0	$\frac{3\alpha}{6-\alpha}$	$\frac{\alpha}{6-\alpha}$	0	0
$z\geq 2$	0	0	0	0	0

Table 5: The values of $f_{22}(0,z,y)$ for $x=0$.

$f_{22}(0,z,y)$	$y=0$	$y=1$	$y=2$	$y\geq 3$
$z=0$	0	$\frac{1-\alpha}{6-\alpha}$	$\frac{1-\alpha}{6-\alpha}$	0
$z=1$	$\frac{\alpha}{6-\alpha}$	$\frac{\alpha}{6-\alpha}$	0	0
$z\geq 2$	0	0	0	0

Table 6: The values of $\psi_i(u)$ for $x=0$.

	$\psi_1(u)$			$\psi_2(u)$		
u	$\alpha=0.8$	$\alpha=0.85$	$\alpha=0.9$	$\alpha=0.8$	$\alpha=0.85$	$\alpha=0.9$
0	0.8500	0.7471	0.6556	1.0000	1.0000	1.0000
1	0.7675	0.6191	0.4948	0.9308	0.8748	0.8176
2	0.6910	0.5110	0.3721	0.8390	0.7232	0.6157
3	0.6223	0.4219	0.2799	0.7554	0.5970	0.4630
4	0.5604	0.3483	0.2105	0.6803	0.4929	0.3483
5	0.5047	0.2876	0.1583	0.6126	0.4070	0.2620
6	0.4545	0.2375	0.1191	0.5517	0.3360	0.1970
7	0.4093	0.1961	0.0896	0.4969	0.2774	0.1482
8	0.3686	0.1619	0.0674	0.4475	0.2291	0.1115
9	0.3320	0.1337	0.0507	0.4030	0.1891	0.0838
10	0.2989	0.1104	0.0381	0.3629	0.1562	0.0631
11	0.2692	0.0911	0.0287	0.3268	0.1289	0.0474
15	0.1771	0.0423	0.0092	0.2150	0.0599	0.0152
20	0.1049	0.0162	0.0022	0.1273	0.0230	0.0037
25	0.0621	0.0062	0.0005	0.0754	0.0088	0.0009
30	0.0368	0.0024	0.0001	0.0447	0.0034	0.0002
35	0.0218	0.0009	0	0.0265	0.0013	0.0001
40	0.0129	0.0004	0	0.0157	0.0005	0
45	0.0076	0.0001	0	0.0093	0.0002	0

From Tables 6-9, we see that the ruin probabilities decrease as α and x increase, this coincides with the intuition. We can also see from the tables that α is quite sensitive to the ruin probabilities for any dividend threshold x, no matter the surplus is small or large. However, for any fixed α, we find that the influence of x on the ruin probabilities

Table 7: The values of $\psi_i(u)$ for $x = 1$.

	$\psi_1(u)$			$\psi_2(u)$		
u	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$
0	0.8235	0.7171	0.6310	1.0000	1.0000	1.0000
1	0.7529	0.6039	0.4833	0.9186	0.8599	0.8046
2	0.6775	0.4982	0.3633	0.8228	0.7053	0.6013
3	0.6102	0.4114	0.2733	0.7406	0.5820	0.4521
4	0.5495	0.3396	0.2055	0.6670	0.4806	0.3401
5	0.4949	0.2804	0.1546	0.6007	0.3968	0.2558
6	0.4456	0.2315	0.1163	0.5410	0.3276	0.1924
7	0.4013	0.1912	0.0875	0.4872	0.2705	0.1447
8	0.3614	0.1578	0.0658	0.4387	0.2233	0.1088
9	0.3255	0.1303	0.0495	0.3951	0.1844	0.0819
10	0.2931	0.1076	0.0372	0.3558	0.1522	0.0616
11	0.2640	0.0888	0.0280	0.3204	0.1257	0.0463
15	0.1736	0.0413	0.0090	0.2108	0.0584	0.0148
20	0.1028	0.0158	0.0022	0.1248	0.0224	0.0036
25	0.0609	0.0061	0.0005	0.0740	0.0086	0.0009
30	0.0361	0.0023	0.0001	0.0438	0.0033	0.0002
35	0.0214	0.0009	0	0.0259	0.0013	0
40	0.0127	0.0003	0	0.0154	0.0005	0
45	0.0075	0.0001	0	0.0091	0.0002	0

Table 8: The values of $\psi_i(u)$ for $x = 2$.

	$\psi_1(u)$			$\psi_2(u)$		
u	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$
0	0.7674	0.6625	0.5910	1.0000	1.0000	1.0000
1	0.6744	0.5275	0.4274	0.8604	0.7975	0.7546
2	0.6185	0.4465	0.3293	0.7602	0.6415	0.5525
3	0.5567	0.3684	0.2475	0.6758	0.5213	0.4096
4	0.5013	0.3042	0.1862	0.6086	0.4304	0.3080
5	0.4514	0.2512	0.1400	0.5481	0.3554	0.2317
6	0.4065	0.2074	0.1053	0.4935	0.2935	0.1742
7	0.3660	0.1713	0.0792	0.4444	0.2423	0.1310
8	0.3296	0.1414	0.0595	0.4002	0.2001	0.0985
9	0.2968	0.1168	0.0448	0.3604	0.1652	0.0741
10	0.2672	0.0964	0.0337	0.3245	0.1364	0.0557
11	0.2406	0.0796	0.0253	0.2922	0.1127	0.0419
15	0.1582	0.0371	0.0081	0.1921	0.0524	0.0134
20	0.0935	0.0143	0.0019	0.1136	0.0202	0.0032
25	0.0553	0.0055	0.0004	0.0672	0.0078	0.0007
30	0.0326	0.0022	0.0001	0.0396	0.0030	0.0001
35	0.0192	0.0009	0	0.0233	0.0012	0
40	0.0112	0.0004	0	0.0137	0.0005	0
45	0.0065	0.0002	0	0.0080	0.0003	0

Table 9: The values of $\psi_i(u)$ for $x = 3$.

	$\psi_1(u)$			$\psi_2(u)$		
u	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.85$	$\alpha = 0.9$
0	0.7041	0.6121	0.5589	1.0000	1.0000	1.0000
1	0.5858	0.4569	0.3825	0.8225	0.7672	0.7353
2	0.5148	0.3638	0.2766	0.6568	0.5500	0.4883
3	0.4722	0.3079	0.2131	0.5803	0.4424	0.3576
4	0.4248	0.2542	0.1602	0.5158	0.3596	0.2651
5	0.3825	0.2099	0.1205	0.4644	0.2969	0.1993
6	0.3444	0.1734	0.0906	0.4182	0.2452	0.1499
7	0.3101	0.1432	0.0681	0.3766	0.2025	0.1128
8	0.2792	0.1183	0.0512	0.3391	0.1672	0.0848
9	0.2514	0.0977	0.0385	0.3053	0.1381	0.0638
10	0.2264	0.0807	0.0290	0.2749	0.1141	0.0480
11	0.2038	0.0667	0.0218	0.2475	0.0942	0.0361
15	0.1339	0.0311	0.0070	0.1626	0.0439	0.0115
20	0.0791	0.0121	0.0017	0.0961	0.0170	0.0028
25	0.0466	0.0048	0.0004	0.0567	0.0067	0.0006
30	0.0274	0.0020	0.0001	0.0334	0.0027	0.0001
35	0.0160	0.0009	0	0.0196	0.0012	0
40	0.0093	0.0005	0	0.0114	0.0006	0
45	0.0053	0.0003	0	0.0065	0.0004	0

is quite small for large surplus. This implies that the insurer who has great wealth would be better to pay dividends as soon as possible.

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